CALCULATION OF TEMPERATURE FIELDS

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A method is discussed for solving heat-transfer problems in various regions for variable thermophysical characteristics. The method is based on the reduction of boundary-value problems of mathematical physics to infinite systems of linear algebraic equations which can be solved by the reduction method.

An increase in the accuracy of thermal calculations requires the development of reliable and rather versatile methods of solving heat-transfer problems. Among the basic heat-transfer problems under consideration are problems of steady-state heat transfer in various regions and for various boundary conditions, steady and unsteady problems with discrete energy sources, nonclassical problems of steady and unsteady heat transfer, problems to be solved by computer, and the determination of the metrological characteristics of the procedure.

§1. We consider a method for solving basic boundary-value problems of steady heat transfer in a rectangular parallelepiped described by elliptic equations.

We consider the following problem in a domain V:

$$L[U(x)] \equiv \sum_{i=1}^{3} \left\{ \frac{\partial}{\partial x_i} \left(\lambda_i(x) \frac{\partial U(x)}{\partial x_i} \right) + b_i(x) \frac{\partial U(x)}{\partial x_i} \right\} = F(x), \tag{1}$$

$$M_{il}[U(x)] = \varphi_{il} \text{ for } x_i = a_{il}, \; i = 1, 2, 3, \; l = 1, 2, \tag{2}$$

where L is an elliptic operator with sufficiently smooth coefficients,

$$M_{il}[U(x)] = \alpha_{il} \frac{\partial U(x)}{\partial x_i} + \beta_{il} U(x),$$

α_{il} and β_{il} are constants, and $\alpha_{il}^2 + \beta_{il}^2 \neq 0$.

Suppose problem (1), (2) has a unique classical solution; i.e., a unique function U(x) exists which is continuous in $\overline{V} = V \cup \Gamma$ and satisfies Eq. (1) in V and boundary conditions (2). We seek the solution of problem (1), (2) in the form

$$U(x) = \vartheta(x) + \Psi(x). \tag{3}$$

We rewrite conditions (2) in the form

$$M_{il}[\vartheta(x)] = 0, \; x \in \Gamma_{il} = \{x_i = a_{il}\}, \tag{4}$$

$$M_{il}[\Psi(x)] = \varphi_{il}, \; x \in \Gamma_{il} = \{x_i = a_{il}\}. \tag{5}$$

By using (3)-(5), Eq. (1) can be written in the form

$$L[\vartheta(x)] = F_1(x), \tag{6}$$

where

$$F_1(x) = F(x) - L[\Psi(x)].$$

We construct the thrice continuously differentiable function $\Psi(x)$ so that it satisfies the inhomogeneous

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conditions (5) and so that the function $F_1(x)$ satisfies the conditions for expansion [1] over the whole region $\bar{V}$ in an absolutely and uniformly convergent series of eigenfunctions of the Laplacian operator in the problem

$$\Delta X(x) + \mu X(x) = 0, \qquad (7)$$
$$M_{il}[X(x)] = 0 \text{ for } x_i = a_{il},$$

where Δ is the Laplacian operator and $\mu > 0$. This requires

$$M_{il}[F_1(x)] = 0, \; x_i = a_{il}. \qquad (8)$$

The function $\Psi(x)$ can be constructed [2] so that $\omega(x) = 0$ is the equation of the surface Γ normalized on Γ to the k-th order.

We seek the solution of problem (6), (4) in the form

$$\vartheta(x) = \eta(x) \sum_{n=1}^{\infty} A_n X_n(x), \qquad (9)$$

where the series is written in the natural order of increasing eigenvalues μ_n of the Laplacian operator in problem (7).

The $\{X_n(x)\}_{n=1}^{\infty}$ form a complete orthogonal set, all the eigenvalues μ_n of problem (8) are nonnegative, and $\mu_n \to +\infty$ as $n \to +\infty$ [3].

The function $\eta(x)$ is constructed so as to satisfy the conditions for expansion of the functions $L[\eta(x)x_n(x)]$ in series of the eigenfunctions of the Laplacian operator which are absolutely and uniformly convergent in the whole domain $\bar{V}$ for all $n = 1, 2, \ldots.$

In order for (9) to satisfy the boundary conditions (4) for the second and third boundary-value problems, it is necessary that

$$\frac{\partial \eta(x)}{\partial x_i} = 0 \text{ for } x_i = a_{il}. \qquad (10)$$

Substituting (9) into (6), formally carrying out all the operations, and expanding both sides of the relation obtained in series of eigenfunctions of the Laplacian operator, which is possible because of the way the functions $\Psi(x)$ and $\eta(x)$ were chosen, we obtain

$$\sum_{n=1}^{\infty} p_{mn} A_n = q_m, \; m = 1, 2, 3, \ldots, \qquad (11)$$

where

$$\begin{cases} p_{mn} = \int\limits_V L[\eta(x) X_n(x)] X_m(x)\, dx, \\ q_m = \int\limits_V F_1(x) X_m(x)\, dx. \end{cases} \qquad (12)$$

Let

$$A_n = \mu_n^{-\alpha} B_n. \qquad (13)$$

We consider the series

$$J_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |p_{mn} \mu_n^{-\alpha}|. \qquad (14)$$

Since the coefficients p_{mn} are the expansion coefficients of the functions $L[\eta(x)X_n(x)]$, $n = 1, 2, \ldots,$ which satisfy the expansion conditions, then by using [1], the Hölder inequality, and the fact that

$$\sum_{m=1}^{\infty} \mu_m^{-2} < +\infty \text{ according to [4]},$$

$$\max |X_n(x)| = O(\mu_n^{\frac{3}{4}}) \text{ according to [5]},$$

we find $J_0 < +\infty$ if

$$\alpha > 3 + \varepsilon, \qquad (15)$$

where ε is an arbitrarily small positive number.

Consequently, the conditions of [6] are satisfied and there is a unique solution of system (11), taking account of (13), satisfying the condition

$$\sum_{n=1}^{\infty} B_n^2 < R < +\infty, \tag{16}$$

which can be found by one of the versions of the reduction method.

The difference between the approximate solution $B^{(N)}$ and the exact solution is estimated as

$$\|PB - PB^{(N)}\| = O(\|P - P_N\| + \|Q - Q_N\|, \tag{17}$$

where

$$P = (p_{mn}\mu_n^{-\alpha})_{m,n=1}^{\infty}, \quad Q = (q_m)_{m=1}^{\infty},$$

$$P_N = \begin{pmatrix} p_{11}\mu_1^{-\alpha} \dots p_{1N}\mu_N^{-\alpha} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ p_{N1}\mu_1^{-\alpha} \dots p_{NN}\mu_N^{-\alpha} & 0 & 0 & \dots \\ 0 \quad \dots \quad 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}; \quad Q_N = \begin{pmatrix} q_1 \\ q_2 \\ \dots \\ q_N \\ 0 \\ 0 \\ \dots \end{pmatrix}.$$

Consequently, the solution of a heat-transfer problem is reduced to the solution of a finite algebraic system whose order depends on the accuracy demanded of the solution of the original problem.

We demonstrate the validity of all the formal operations and the fact that (9) is the classical solution of problem (1), (2) by proving the uniform convergence of the series

$$J_1 = \sum_{n=1}^{\infty} |A_n X_n(x)|, \quad J_2 = \sum_{n=1}^{\infty} \left| A_n \frac{\partial X_n(x)}{\partial x_i} \right|,$$

$$J_3 = \sum_{n=1}^{\infty} \left| A_n \frac{\partial^2 X_n(x)}{\partial x_i \partial x_j} \right|,$$

where $i, j = 1, 2, 3$.

Using the Hölder inequality and [7] we prove the convergence of the series J_1, J_2, J_3.

To solve problem (1), (2) it is necessary to find the eigenvalues and eigenfunctions of the Laplacian operator in problem (7), to construct the auxiliary functions $\eta(x)$ and $\Psi(x)$ by using [2], and to calculate the coefficients p_{mn} and q_m as functions of the form of the operator L and the function $F_1(x)$. Then, depending on the accuracy demanded of the solution of problem (1), (2), it is necessary to choose the order of the finite system and to solve it. We write the solution of problem (1), (2) in the form (3). The rate of convergence of the series in (3) depends on the smoothness of the coefficients in Eq. (1) and on the smoothness of the functions F(x) and φ_{il}, where $i = 1, 2, 3$; $l = 1, 2$.

§2. The shapes and complexity of the objects considered create certain difficulties in thermal calculations. The simplification and subdivision of complex regions introduce additional errors which show up particularly in high-temperature processes.

Let us consider the first boundary-value problem in a domain W with a sufficiently smooth boundary Γ:

$$L[U(x)] = F(x), \tag{18}$$

$$U(x) = \varphi(x), \quad x \in \Gamma. \tag{19}$$

We assume that the coefficients in Eq. (1) and the functions F(x) and $\varphi(x)$ are sufficiently smooth and that problem (18), (19) has a unique classical solution which we seek in the form (3).

The function $\Psi(x)$ is twice continuously differentiable and satisfies conditions (19); i.e., we construct

$$\Psi(x) = \varphi(x) \text{ for } x \in \Gamma, \tag{20}$$

so as to satisfy the condition

$$F_1(x) = 0 \text{ for } x \in \Gamma, \tag{21}$$

where

$$F_1(x) = F(x) - L[\Psi(x)].$$

To find the function $\vartheta(x)$ we obtain the boundary-value problem

$$L[\vartheta(x)] = F_1(x), \tag{22}$$

$$\vartheta(x) = 0 \quad \text{for} \quad x \in \Gamma. \tag{23}$$

We imbed the domain $\overline{W} = W \cup \Gamma$ in a rectangular parallelepiped V and use the fictitious domain theory [8] to treat in V the problem obtained from (22) and (23):

$$L[\vartheta_1(x)] = F_1(x), \; x \in W, \tag{24}$$

$$D\Delta[\vartheta_1(x)] = 0, \quad x \in V \setminus \overline{W}, \tag{25}$$

$$\vartheta_1(x) = 0, \; x \in \Gamma_0, \tag{26}$$

$$\vartheta_1(x)|_{\Gamma+} = \vartheta_1(x)|_{\Gamma-}, \tag{27}$$

$$\left.\frac{d\vartheta_1(x)}{dN}\right|_{\Gamma+} = \left.\frac{d\vartheta_1(x)}{dN}\right|_{\Gamma-} D, \tag{28}$$

where Γ_0 is the boundary of domain V, D is some large positive number, Γ^+ and Γ^- are, respectively, the inside and outside of the boundary Γ, d/dN is the normal derivative, and [8]

$$\|\vartheta(x) - \vartheta_1(x)\|_{L_2(W)} \leqslant CD^{-\frac{1}{2}} \|F_1(x)\|_{L_2(W)},$$

where C is a constant independent of ϑ, ϑ_1, and D.

Let

$$\vartheta_1(x) = \begin{cases} \omega(x) \sum_{n=1}^{\infty} A_n X_n(x), & x \in W, \\ D^{-1}\omega(x)\eta_1(x) \sum_{n=1}^{\infty} A_n X_n(x), & x \in V \setminus W, \end{cases} \tag{29}$$

where $\omega(x)$ is such that the equation $\omega(x) = 0$ is the equation of the surface Γ, and the series in (29) are written in the natural order of increasing eigenvalues of problem (7), where $\alpha_{il} = 0$ for all i = 1, 2, 3, and l = 1, 2.

We seek the function $\eta_1(x)$ from the continuity condition for the function

$$f_1(x) = \begin{cases} \sum_{n=1}^{\infty} A_n g_{1n}(x), & x \in W, \\ \sum_{n=1}^{\infty} A_n g_{2n}(x), & x \in V \setminus W, \end{cases} \tag{30}$$

where

$$g_{1n}(x) = L[\omega(x) X_n(x)]; \; g_{2n}(x) = \Delta[\omega(x)\eta_1(x) X_n(x)],$$

and conditions (28) are satisfied.

The properties of the functions $X_n(x)$ and $\omega(x)$ ensure that the boundary conditions (26), (27) are satisfied automatically. Consequently, for (29) to be a solution of problem (18), (19) it is necessary that

$$f_1(x) = f_2(x), \; \forall x \in V, \tag{31}$$

where

$$f_2(x) = \begin{cases} F_1(x) & \text{for } x \in W, \\ 0 & \text{for } x \in V \setminus W. \end{cases}$$

Expanding both sides of Eq. (31) in series of the eigenfunctions of the Laplacian operator, which is possible because of the structure of the function $\vartheta_1(x)$, we obtain an infinite set of linear algebraic equations (11) whose coefficients have the form

$$p_{mn} = \int_V g_n(x) X_m(x)\,dx,$$

$$q_m = \int_V f_2(X) X_m(x)\,dx, \quad (32)$$

where

$$g_n(x) = \begin{cases} g_{1n}(x) & \text{for } x \in W, \\ g_{2n}(x) & \text{for } x \in V \setminus W. \end{cases}$$

The proofs of the applicability of the reduction method for solving system (11) with the coefficients (32) and the convergence of the series are similar to those in Sec. 1.

The second and third boundary-value problems are solved by reducing them to problem (18), (19) [9]. In this case an additional system analogous to (11) must be solved, but this presents no difficulty when a computer is used.

LITERATURE CITED

1. V. A. Il'in, Usp. Mat. Nauk, 13, No. 1 (1958).
2. V. L. Rvachev and A. P. Slesarenko. Algebra of Logic and Integral Transforms in Boundary-Value Problems [in Russian], Naukova Dumka, Kiev (1976).
3. V. S. Vladimirov, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1967).
4. É. C. Titchmarsh, Eigenfunction Expansions Associated with Second Order Differential Equations, Clarendon, Oxford (1962).
5. V. A. Il'in and I. A. Shishmarev, in: Materials for a Joint Soviet—American Symposium on Partial Differential Equations [in Russian], Novosibirsk (1963).
6. B. G. Gabdulkhaev, in: Scientific Proceedings [in Russian], Vol. 128, Book 5, Izd. Kazansk. Univ., Kazan' (1968).
7. V. A. Il'in, Usp. Mat. Nauk, 15, No. 2 (1960).
8. V. K. Saul'ev, Sibirsk. Mat. Zh., 4, No. 2 (1964).
9. O. M. Kozlov, Ukrainsk. Mat. Zh., 16, No. 2 (1964).